ERROR ANALYSIS OF A FRACTIONAL TIME-STEPPING TECHNIQUE FOR INCOMPRESSIBLE FLOWS WITH VARIABLE DENSITY

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ABSTRACT. In this paper we analyze the convergence properties of a new fractional time-stepping technique for the solution of the variable density incompressible Navier-Stokes equations. The main feature of this method is that, contrary to other existing algorithms, the pressure is determined by just solving one Poisson equation per time step. The method is formally second-order accurate in time. Stability and error estimates are proven.

1. INTRODUCTION

A fractional time stepping-technique for solving incompressible viscous flows with variable density is introduced and analyzed in this paper. The fluid flows in question are governed by the time-dependent Navier-Stokes equations:

$$\begin{cases} \rho_t + \nabla\cdot(\rho u) = 0, \\ \rho\left(u_t + u\cdot\nabla u\right) + \nabla p - \mu\Delta u = f, \\ \nabla\cdot u = 0, \end{cases} \tag{1.1}$$

where the independent variables are the density $\rho > 0$, the velocity field u, and the pressure p. The constant $\mu > 0$ is the dynamic viscosity coefficient and f is a driving external force. The fluid occupies a bounded domain Ω in $\mathbb{R}^d$ (with $d = 2$ or 3) and a solution to the above problem is considered over the time interval $[0, T]$. The system (1.1) is supplemented with the following initial and boundary conditions for the density and velocity:

$$\begin{cases} \rho(x,0) = \rho_0(x), \quad \rho(x,t)|_{\Gamma^-} = a(x,t), \\ u(x,0) = u_0(x), \quad u(x,t)|_{\Gamma} = b(x,t), \end{cases} \tag{1.2}$$

where $\Gamma = \partial\Omega$ and Γ^- is the inflow boundary, which is defined by $\Gamma^- = \{x \in \Gamma : \ u\cdot n < 0\}$, where n is the outward unit normal vector. Throughout this paper we assume that the boundary is impermeable, i.e., $u\cdot n = 0$ everywhere on Γ, so that $\Gamma^- = \emptyset$.

Approximating (1.1) and (1.2) can be done by solving the coupled system (1.1), but this approach may sometimes be computer intensive due to the saddle point structure of the problem. We refer to [25] where such a strategy is developed. Alternative and more efficient approaches usually advocated in the literature consist of using factional time-stepping strategies and exploiting, as far as possible, the techniques and results already established for the solution of constant density incompressible

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fluid flows. The starting point of most fractional time-stepping algorithms is Chorin's [7] and Temam's [27] projection method which consists of decoupling the incompressibility constraint and the diffusion. Several algorithms extending this idea to variable density fluid flows have been proposed in the literature, see for example [1, 4, 18, 26]. However, to the best of our knowledge, [18, 26] are the only papers where projection methods for variable density flows have been proved to be stable and no rigorous error analysis of these methods has been done yet. Moreover, a common feature of all the projection-like methods referred to above is that at each time step, say t^{k+1}, the pressure or some related scalar unknown, say Φ, must be computed by solving an equation of the following form:

$$-\nabla\cdot\left(\frac{1}{\rho^{k+1}}\nabla\Phi\right) = \Psi, \quad \partial_n\Phi|_\Gamma = 0,$$

where ρ^{k+1} is an approximation of the density at time t^{k+1} and Ψ is a right-hand side that varies at each time step. Solving efficiently this problem if far more technical than just solving a Poisson equation as it requires assembling and preconditioning a variable-coefficient stiffness matrix at each time step. Note also in passing that it is necessary to have a uniform lower bound on the density for this problem to be solvable. This condition is often overlooked in the literature.

On the basis of the observations above, we have recently started a research program whose objective is to develop a fractional time-stepping strategy that only requires the solution of a Poisson problem to compute the pressure, [19, 21]. The stability of a first-order variant of the method was proven in [19, 21], but the error analysis was missing and whether second-order variants of the proposed method could be proven to be stable was still an open question. The goal of the present paper is to fill these two gaps. First we provide a rigorous error analysis for the first-order method introduced in [19, 21]. We prove that, provided the density equation is solved correctly, the accuracy of our fractional time-stepping technique is as good as the corresponding schemes for constant density flows. Second, we introduce a second-order version of the method and we prove its stability.

The paper is organized as follows. Notation along with space and time discretizations is introduced in Section 2. The first-order algorithm is described in Section 3. The error analysis of this algorithm is done is Section 4. We show that, provided the density equation is correctly approximated, the method performs as well as its constant density counterpart. A formally second-order version of the algorithm is introduced in Section 5 and the stability of the method is proven. As a byproduct of our analysis we are able to provide a new proof of stability of the second-order incremental pressure-correction scheme in standard form (see [15]). The novelty of our analysis is that we eliminate the so-called projected velocity from the algorithm. Numerical experiments illustrating the performance of the method are reported in Section 6.

2. Preliminaries

2.1. Notation and Assumptions.

We consider the time-dependent variable density Navier-Stokes system (1.1)-(1.2) on the finite time interval $[0,T]$ and in an open connected and bounded domain $\Omega\subset\mathbb{R}^d$ ($d = 2$ or 3) with boundary Γ, which we assume to be sufficiently smooth. More precisely, we assume that Ω is such that the Stokes operator possesses the usual regularization properties (see [6, 28]).

Let $\tau > 0$ be a time step and let us set $t_k = k\tau$ for $0 \le k \le K := [T/\tau]$. Let E be a normed space equipped with the norm $\|\cdot\|_E$. For any time-dependent function $\phi : [0,T] \longrightarrow E$, we denote $\phi^k := \phi(t^k)$, and the sequence $\phi^0, \phi^1, \ldots, \phi^K$ is denoted ϕ_τ. To simplify the notation we define the time-increment operator δ by setting

$$\delta\phi^k = \phi^k - \phi^{k-1}, \tag{2.1}$$

and we define the following discrete norms:

$$\|\phi_\tau\|_{\ell^2(E)} := \left(\tau \sum_{k=0}^{K} \|\phi^k\|_E^2\right)^{1/2}, \qquad \|\phi_\tau\|_{\ell^\infty(E)} := \max_{0\le k\le K}\left(\|\phi^k\|_E\right). \tag{2.2}$$

The space of functions $\phi : [0,T] \longrightarrow E$ such that the map $(0,T) \ni t \longrightarrow \|\phi(t)\|_E \in \mathbb{R}$ is L^p-integrable is indifferently denoted $L^p((0,T);E)$ or $L^p(E)$.

No notational distinction is done between scalar or vector-valued functions but spaces of vector-valued functions are identified with bold fonts. The space of functions in $L^2(\Omega)$ that have zero average is denoted $L^2_0(\Omega)$. We use the standard Sobolev spaces $W^{m,p}(\Omega)$, for $0 \le m \le \infty$ and $1 \le p \le \infty$. The closure with respect to the norm $\|\cdot\|_{W^{m,p}}$ of the space of $\mathcal{C}^\infty$-functions compactly supported in Ω is denoted $W^{m,p}_0(\Omega)$. To simplify the notation, the Hilbert space $W^{s,2}(\Omega)$ (resp. $W^{s,2}_0(\Omega)$) is denoted $H^s(\Omega)$ (resp. $H^s_0(\Omega)$). The scalar product of $L^2(\Omega) := H^0(\Omega)$ is denoted $\langle\cdot,\cdot\rangle$.

Henceforth c denotes a generic constant whose value may change at each occurrence. This constant may depend on the data of the problem and its exact solution, but it does not depend on the discretization parameters or the solution of the numerical scheme.

2.2. **The Space Discretization.** To construct a Galerkin approximation of (1.1)–(1.2) we introduce three sequences of finite-dimensional spaces $\{W_h\}_{h>0}$, $\{\mathbf{X}_h\}_{h>0}$, $\{M_h\}_{h>0}$, for $h > 0$, with $W_h \subset H^1(\Omega)$, $\mathbf{X}_h \subset \mathbf{H}^1_0(\Omega)$ and $M_h \subset H^1(\Omega)$. We use W_h, $\mathbf{X}_h$, and M_h to approximate the density, the velocity, and the pressure, respectively. We assume that the pair of spaces $(\mathbf{X}_h, M_h)$ satisfies a discrete inf–sup condition (cf. [10, 9]), i.e., there is $c > 0$ independent of h such that

$$\inf_{q_h\in M_h}\sup_{v_h\in\mathbf{X}_h}\frac{\int_\Omega v_h\cdot\nabla q_h}{\|q_h\|_{L^2}\|v_h\|_{\mathbf{H}^1}} \ge c.$$

Moreover, we assume that the following approximation properties hold (cf. [10, 9]): There is $l \in \mathbb{N}$ such that for all $\ell \in [0,l]$

$$\inf_{r_h\in W_h}\|r-r_h\|_{L^2} \le ch^{\ell+1}\|r\|_{H^{\ell+1}}, \qquad \forall r\in H^{\ell+1}(\Omega). \tag{2.3}$$

$$\inf_{v_h\in\mathbf{X}_h}\{\|v-v_h\|_{\mathbf{L}^2} + h\|v-v_h\|_{\mathbf{H}^1}\} \le ch^{\ell+1}\|v\|_{\mathbf{H}^{\ell+1}}, \qquad \forall v\in\mathbf{H}^{\ell+1}(\Omega)\cap\mathbf{H}^1_0(\Omega), \tag{2.4}$$

$$\inf_{q_h\in M_h}\|q-q_h\|_{L^2} \le ch^{\ell}\|q\|_{H^\ell}, \qquad \forall q\in H^\ell(\Omega)\cap L^2_0(\Omega). \tag{2.5}$$

For any t in $[0,T]$ we define the Stokes projection of the solution $(u(t),p(t))$ of (1.1)–(1.2) as the pair $(w_h(t),q_h(t)) \in \mathbf{X}_h\times M_h$ that solves

$$\begin{cases}\langle\nabla w_h(t),\nabla v_h\rangle + \langle\nabla q_h(t),v_h\rangle = \langle\nabla u(t),v_h\rangle - \langle p(t),\nabla\cdot v_h\rangle, & \forall v_h\in\mathbf{X}_h,\\ \langle w_h(t),\nabla r_h\rangle = 0, & \forall r_h\in M_h.\end{cases} \tag{2.6}$$

Owing to the regularization properties of the Stokes operator, the following estimates hold:

Lemma 2.1. *If $u \in L^\beta\left(\mathbf{H}^{l+1}(\Omega)\cap\mathbf{H}^1_0(\Omega)\right)$ and $p \in L^\beta\left(H^l(\Omega)\right)$ for $1\le\beta\le\infty$, then there exists $c>0$ such that*

$$\begin{aligned}\|u-w_h\|_{L^\beta(\mathbf{L}^2(\Omega))} + h\left[\|u-w_h\|_{L^\beta(\mathbf{H}^1(\Omega))} + \|p-q_h\|_{L^\beta(L^2(\Omega))}\right]&\\ \le ch^{l+1}\left[\|u\|_{L^\beta(\mathbf{H}^{l+1}(\Omega))} + \|p\|_{L^\beta(H^l(\Omega))}\right].&\end{aligned} \tag{2.7}$$

Moreover, if $u \in L^\beta\left(\mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)\right)$ *and* $p \in L^\beta\left(H^1(\Omega)\right)$

$$\|w_h\|_{L^\beta(\mathbf{L}^\infty(\Omega)\cap\mathbf{W}^{1,3}(\Omega))} + \|q_h\|_{L^\beta(H^1(\Omega))} \leq c\left[\|u\|_{L^\beta(\mathbf{H}^2(\Omega))} + \|p\|_{L^\beta(H^1(\Omega))}\right]. \tag{2.8}$$

3. Description of the First-Order Scheme

We now describe the first-order fractional time-stepping scheme as introduced in [19, 21]. We refer the reader to [19, 21] for the heuristics behind the algorithm.

3.1. Initialization.

Given the initial data (ρ_0, u_0), we construct the approximate data $(\rho_h^0, u_h^0, p_h^0) \in W_h \times \mathbf{X}_h \times M_h$ so that

$$\|\rho_0 - \rho_h^0\|_{L^\infty} + \|u_0 - u_h^0\|_{\mathbf{L}^2} + h\|u_0 - u_h^0\|_{\mathbf{H}^1} + h\|p_0 - p_h^0\|_{L^2} \leq ch^{l+1}. \tag{3.1}$$

The initial pressure p_h^0 can be computed from the pair (ρ_0, u_0), see [15] for more details.

We henceforth assume that $\min_{x\in\bar\Omega} \rho_0(x) > 0$ and that the approximate density field ρ_h^0 satisfies the following property

$$\chi \leq \rho_h^0 \leq \varrho, \tag{3.2}$$

where the parameters χ and ϱ are assumed to satisfy the following property

$$\chi \leq \min_{x\in\bar\Omega} \rho_0(x), \qquad \sup_{x\in\bar\Omega} \rho_0(x) \leq \varrho. \tag{3.3}$$

The role of the parameters χ and ϱ is clarified in the next sub-section.

3.2. Time-Stepping Technique.

Given $(\rho_h^k, u_h^k, p_h^k) \in W_h \times \mathbf{X}_h \times M_h$ we now describe how to obtain the next approximations $(\rho_h^{k+1}, u_h^{k+1}, p_h^{k+1}) \in W_h \times \mathbf{X}_h \times M_h$. The algorithm proceeds in three steps: (i) density update, (ii) velocity update, (iii) pressure update.

3.2.1. *Density Update.* The density update is computed using the mass conservation equation, which we recall is hyperbolic. It is well known that Galerkin techniques are not well suited for the solution of hyperbolic problems (see for instance [9]). The list of techniques aiming at addressing this issue is endless; among these methods one can cite Galerkin-Least Squares [23], Discontinuous-Galerkin [23, 30], subgrid viscosity [14], method of characteristics [8], edge stabilization [5], entropy viscosity [16] and many others. We assume that the sequence of approximate densities $\{\rho_h^k\}_{k=0,\dots,K} \subset W_h$ is obtained by one of these stabilization techniques. More precisely, we assume that given the pair $(\rho_h^k, u_h^k) \in W_h \times \mathbf{X}_h$, the approximation technique that is used to approximate the mass conservation returns ρ_h^{k+1} and that this algorithm satisfies the following stability hypothesis:

$$\chi \leq \min_{x\in\bar\Omega} \rho_h^{k+1}(x), \qquad \sup_{x\in\bar\Omega} \rho_h^{k+1} \leq \varrho, \qquad \forall k \geq 1. \tag{3.4}$$

Note that this is a natural assumption since, owing to the incompressibility of the velocity field, the density field ρ satisfies the following properties: $\rho(t) \in [\min_{x\in\bar\Omega} \rho_0(x), \sup_{x\in\bar\Omega} \rho_0(x)]$ for all $t \geq 0$, cf. Lions [24]. For instance, first-order monotone schemes satisfy (3.4) with $\chi = \min_{x\in\bar\Omega} \rho_0(x)$ and $\varrho = \sup_{x\in\bar\Omega} \rho_0(x)$.

3.2.2. *Velocity Update.* Having obtained an approximate density, we define

$$\rho_h^\star := \frac{1}{2}\left(\rho_h^{k+1} + \rho_h^k\right), \tag{3.5}$$

$$p_h^\sharp := p_h^k + \gamma\delta p_h^k, \quad \gamma \in \{0,1\}. \tag{3.6}$$

The parameter γ is user-dependent. We say that the method is non-incremental if $\gamma = 0$ and incremental if $\gamma = 1$. The incremental version of the algorithm is more accurate that the non-incremental one. We just mention the non-incremental version of the algorithm for historical reasons: the original algorithm of Chorin and Temam for constant density incompressible flows is non-incremental [7, 27]. When $\gamma = 1$, we take $\delta p_h^0 = 0$.

The next approximation of the velocity field $u_h^{k+1} \in \mathbf{X}_h$ is computed by solving the following problem:

$$\begin{aligned} &\left\langle \frac{\rho_h^\star u_h^{k+1} - \rho_h^k u_h^k}{\tau}, v_h \right\rangle + \left\langle \rho_h^{k+1} u_h^k \cdot \nabla u_h^{k+1}, v_h \right\rangle \\ &\quad + \left\langle \tfrac{1}{2}\nabla\cdot\left(\rho_h^{k+1}u_h^k\right)u_h^{k+1}, v_h\right\rangle + \mu\left\langle \nabla u_h^{k+1}, \nabla v_h\right\rangle + \left\langle \nabla p_h^\sharp, v_h\right\rangle = \left\langle f^{k+1}, v_h\right\rangle, \ \forall v_h \in \mathbf{X}_h. \end{aligned} \tag{3.7}$$

3.2.3. *Pressure Update.* Finally, we define the pressure approximation $p_h^{k+1} \in M_h$. First, we let $\phi_h^\flat \in M_h$ be the solution of:

$$\left\langle \nabla\phi_h^\flat, \nabla r_h \right\rangle = \frac{\chi}{\tau}\left\langle u_h^{k+1}, \nabla r_h\right\rangle, \ \forall r_h \in M_h, \tag{3.8}$$

then we set

$$p_h^{k+1} = \phi_h^\flat + \gamma p_h^k, \quad \gamma \in \{0,1\}. \tag{3.9}$$

Remark 3.1. The algorithm (3.5)–(3.9) unifies the non-incremental ($\gamma = 0$) and incremental ($\gamma = 1$) first-order schemes of [19, 21]. The non-incremental version presented here is a slight simplification of the one presented in the references above.

Remark 3.2. One remarkable feature of the algorithm (3.5)–(3.9) is that, apart from condition (3.4), nothing else is required on the sequence of approximate densities to prove stability.

Remark 3.3. Let us introduce the auxiliary space $\mathbf{Y}_h := \mathbf{X}_h + \nabla M_h$. In view of (3.8), the quantity

$$\bar{u}_h^k := u_h^k - \frac{\tau}{\chi}\nabla\phi_h^\flat \in \mathbf{Y}_h,$$

is discretely divergence free (in the sense that $\langle \bar{u}_h^k, \nabla r_h\rangle = 0$ for all $r_h \in M_h$) and could be used as a solenoidal approximation of the velocity. This particular choice of $\mathbf{Y}_h$ fits into de commutative diagram framework described in [11, 12, 17]. Therefore, it could be possible to develop a much more general theory about fractional time-stepping techniques for variable density incompressible flows that would include our method as a particular instance. More specifically, let us assume that one has at hand a space $\mathbf{Y}_h$ so that $\mathbf{X}_h \subset \mathbf{Y}_h$. Let $B_h : \mathbf{X}_h \longrightarrow M_h$ be the operator defined by $\langle Bv_h, q_h\rangle := \langle \nabla\cdot v_h, q_h\rangle$ for all $v_h \in \mathbf{X}_h$ and all q_h in M_h. Assume that one can construct an extension of B_h over $\mathbf{Y}_h$, say $C_h : \mathbf{Y}_h \longrightarrow M_h$. The operator C_h being an extension of B_h over $\mathbf{Y}_h$ means that $B_h = C_h i_h$, where i_h is the natural injection $i_h : \mathbf{X}_h \longrightarrow \mathbf{Y}_h$. Then, in this setting our theory will work by replacing (3.8) by

$$C_h C_h^T \phi_h^\flat = \frac{\chi}{\tau} B_h u_h^{k+1}. \tag{3.10}$$

We leave the details to the reader.

4. Error Estimates for the First-Order Scheme

We prove in this section that the algorithm (3.5)–(3.9) is stable and convergent.

4.1. **Consistency analysis.** To simplify the notation, we introduce the following functions to represent the errors:

$$\begin{cases} \eta(t) := u(t) - w_h(t), & \mu(t) := p(t) - q_h(t), \\ e_h^k := w_h^k - u_h^k, & \epsilon_h^k := q_h^k - p_h^k, \end{cases} \tag{4.1}$$

The functions $\eta(t)$ and $\mu(t)$ can be regarded as the interpolation errors, whereas the functions e_h^k and ϵ_h^k represent the approximation errors. In addition to (3.1), we make the following regularity assumptions on the exact solution of problem (1.1):

$$\rho \in W^{1,\infty}\left(W^{1,\infty}(\Omega)\right), \quad u \in W^{1,\infty}\left(\mathbf{H}_0^1(\Omega) \cap \mathbf{H}^{l+1}(\Omega)\right), \quad p \in W^{1,\infty}\left(H^l(\Omega)\right). \tag{4.2}$$

Let us now determine the equations that control the errors. By taking the difference between the first equation of (2.6) and (3.7) we obtain the equation that controls e_h^k:

$$\left\langle \frac{\rho_h^\star e_h^{k+1} - \rho_h^k e_h^k}{\tau}, v_h \right\rangle + \mu \left\langle \nabla e_h^{k+1}, \nabla v_h \right\rangle + \left\langle \nabla \left(q_h^{k+1} - p_h^\sharp \right), v_h \right\rangle = \mathcal{R}^{k+1}(v_h), \quad \forall v_h \in \mathbf{X}_h, \tag{4.3}$$

where the residual $\mathcal{R}^{k+1}(v_h)$ is decomposed as follows

$$\mathcal{R}^{k+1}(v_h) = R_0^{k+1}(v_h) + R_1^{k+1}(v_h) + R_{nl}^{k+1}(v_h), \tag{4.4}$$

and

$$R_0^{k+1}(v_h) := \left\langle \rho_h^k \frac{w_h^{k+1} - w_h^k}{\tau} - \rho^{k+1} u_t^{k+1}, v_h \right\rangle, \tag{4.5}$$

$$R_1^{k+1}(v_h) := \frac{1}{2} \left\langle \frac{\rho_h^{k+1} - \rho_h^k}{\tau} w_h^{k+1} - \rho_t^{k+1} u^{k+1}, v_h \right\rangle, \tag{4.6}$$

$$R_{nl}^{k+1}(v_h) := \left\langle \rho_h^{k+1} u_h^k \cdot \nabla u_h^{k+1} - \rho^{k+1} u^{k+1} \cdot \nabla u^{k+1}, v_h \right\rangle \tag{4.7}$$

$$+ \frac{1}{2} \left\langle \nabla \cdot (\rho_h^{k+1} u_h^k) u_h^{k+1} - \nabla \cdot (\rho^{k+1} u^{k+1}) u^{k+1}, v_h \right\rangle. \tag{4.8}$$

To obtain the equation that controls the quantity ϵ_h^k we use (3.8) along with the property that $\langle w_h, \nabla r_h \rangle = 0$ for all $r_h \in M_h$,

$$\left\langle \nabla \epsilon_h^\flat, \nabla r_h \right\rangle = \frac{\chi}{\tau} \left\langle e_h^{k+1}, \nabla r_h \right\rangle + \left\langle \nabla q_h^\flat, \nabla r_h \right\rangle, \tag{4.9}$$

where for any sequence ψ_τ we henceforth denote

$$\psi^\flat = \psi^{k+1} - \gamma \psi^k, \qquad \text{and} \qquad \psi^\sharp = \psi^k + \delta \psi^k. \tag{4.10}$$

The two equations (4.3)-(4.9) will be used repeatedly in the error analysis.

The error analysis is based on energy arguments and the first of these arguments consists of testing (4.3) with $v_h := 2\tau e_h^{k+1}$. Then, upon observing that $2e_h^{k+1}(\rho_h^\star e_h^{k+1} - \rho_h^k e_h^k) = \rho_h^{k+1}(e_h^{k+1})^2 + \rho_h^k(\delta e_h^{k+1})^2 - \rho_h^k(e_h^k)^2$, (see e.g. [19, Theorem 3.1]), testing (4.3) with $v_h := 2\tau e_h^{k+1}$ gives

$$\begin{aligned} &\|\sigma_h^{k+1} e_h^{k+1}\|_{\mathbf{L}^2}^2 - \|\sigma_h^k e_h^k\|_{\mathbf{L}^2}^2 + \|\sigma_h^k \delta e_h^{k+1}\|_{\mathbf{L}^2}^2 + 2\mu\tau \left\| e_h^{k+1} \right\|_{\mathbf{H}^1}^2 \\ &\qquad + 2\tau \left\langle \nabla \epsilon_h^\sharp, e_h^{k+1} \right\rangle = 2\tau \left\langle \nabla (q_h^\sharp - q_h^{k+1}), e_h^{k+1} \right\rangle + 2\tau \mathcal{R}^{k+1}(e_h^{k+1}), \end{aligned} \tag{4.11}$$

where we have introduced the notation $\sigma_h := \sqrt{\rho_h}$.

We finish this section by giving an estimate on the consistency residual $2\tau\mathcal{R}^{k+1}(e_h^{k+1})$; the following Lemma provides this estimate.

Lemma 4.1. *Assume that the solution to* (1.1)-(1.2) *satisfies* (4.2) *and that the sequence of approximate densities* $\{\rho_h^k\}$ *satisfies* (3.4). *Then*

$$
|\mathcal{R}^{k+1}(e_h^{k+1})| \leq c\left[\tau + h^{l+1} + \|\rho_h^k - \rho^k\|_{L^2} + \left\|\frac{1}{\tau}\delta\rho_h^{k+1} - \rho_t^{k+1}\right\|_{L^2} + \|\rho_h^{k+1} - \rho^{k+1}\|_{H^1}\right]^2 + \frac{1}{2}\mu\|e_h^{k+1}\|_{\mathbf{H}^1}^2 + c\|\sigma_h^k e_h^k\|_{\mathbf{L}^2}^2. \tag{4.12}
$$

Proof. We estimate separately each of the terms that compose $\mathcal{R}^{k+1}(e_h^{k+1})$. For the first term

$$
\begin{aligned}
R_0^{k+1}(e_h^{k+1}) &= \left\langle \rho_h^k \frac{1}{\tau}\delta w_h^{k+1} - \rho^{k+1}u_t^{k+1}, e_h^{k+1}\right\rangle \\
&= \left\langle \rho_h^k\left(\frac{1}{\tau}\delta w_h^{k+1} - u_t^{k+1}\right), e_h^{k+1}\right\rangle + \left\langle(\rho_h^k - \rho^k)u_t^{k+1}, e_h^{k+1}\right\rangle - \left\langle\delta\rho^{k+1}u_t^{k+1}, e_h^{k+1}\right\rangle \\
&\leq c\|e_h^{k+1}\|_{\mathbf{L}^6}\left(\|\rho_h^k\|_{L^\infty}\left\|\frac{1}{\tau}\delta w_h^{k+1} - u_t^{k+1}\right\|_{\mathbf{L}^2} + (\|\rho_h^k - \rho^k\|_{L^2} + \|\delta\rho^{k+1}\|_{L^2})\|u_t^{k+1}\|_{\mathbf{L}^3}\right) \\
&\leq c\|e_h^{k+1}\|_{\mathbf{H}^1}\left(\tau + h^{l+1} + \|\rho_h^k - \rho^k\|_{L^2}\right),
\end{aligned}
$$

where we used (2.7), (3.4), and (4.2) to derive the last inequality.

We proceed similarly for the second term,

$$
\begin{aligned}
R_1^{k+1}(e_h^{k+1}) &= \frac{1}{2}\left\langle\frac{1}{\tau}\delta\rho_h^{k+1}w_h^{k+1} - \rho_t^{k+1}u^{k+1}, e_h^{k+1}\right\rangle \\
&= \frac{1}{2}\left\langle\left(\frac{1}{\tau}\delta\rho_h^{k+1} - \rho_t^{k+1}\right)w_h^{k+1}, e_h^{k+1}\right\rangle + \frac{1}{2}\left\langle\rho_t^{k+1}(w_h^{k+1} - u^{k+1}), e_h^{k+1}\right\rangle \\
&\leq c\|e_h^{k+1}\|_{\mathbf{L}^6}\left(\left\|\frac{1}{\tau}\delta\rho_h^{k+1} - \rho_t^{k+1}\right\|_{L^2}\|w_h^{k+1}\|_{\mathbf{L}^3} + \|\rho_t^{k+1}\|_{L^3}\|w_h^{k+1} - u^{k+1}\|_{\mathbf{L}^2}\right) \\
&\leq c\|e_h^{k+1}\|_{\mathbf{H}^1}\left(h^{l+1} + \left\|\frac{1}{\tau}\delta\rho_h^{k+1} - \rho_t^{k+1}\right\|_{L^2}\right),
\end{aligned}
$$

where we used (2.7), (2.8), and (4.2) to derive the last inequality.

The derivation of an estimate for the nonlinear advection component of the residual is done by repeating an argument from [13]; we slightly modify the argument though to account for the fact that the density is not constant. We begin by noticing that, for functions that are smooth enough for the integrals to make sense, the following identity holds:

$$
\langle\rho u\cdot\nabla v, v\rangle + \frac{1}{2}\langle\nabla\cdot(\rho u)v, v\rangle = 0.
$$

Then, using the above identity with $v = e_h$, we rewrite the term $R_{nl}^{k+1}(e_h^{k+1})$ as follows

$$
\begin{aligned}
R_{nl}^{k+1}(e_h^{k+1}) = &-\left\langle\rho_h^{k+1}e_h^k\cdot\nabla w_h^{k+1} + \tfrac{1}{2}\nabla\cdot(\rho_h^{k+1}e_h^k)w_h^{k+1}, e_h^{k+1}\right\rangle \\
&+ \left\langle(\rho_h^{k+1} - \rho^{k+1})w_h^k\cdot\nabla w_h^{k+1} + \tfrac{1}{2}\nabla\cdot((\rho_h^{k+1} - \rho^{k+1})w_h^k)w_h^{k+1}, e_h^{k+1}\right\rangle \\
&+ \left\langle\rho^{k+1}\left(w_h^k\cdot\nabla w_h^{k+1} - u^{k+1}\cdot\nabla u^{k+1}\right) + \tfrac{1}{2}\left(\nabla\cdot(\rho^{k+1}w_h^k)w_h^{k+1} - \nabla\cdot(\rho^{k+1}u^{k+1})u^{k+1}\right), e_h^{k+1}\right\rangle \\
&:= A_1 + A_2 + A_3
\end{aligned}
$$

Since the approximate density sequence $\{\rho_h^k\}$ satisfies (3.4) and the approximate velocity sequence $\{w_h^k\}$ satisfies (2.8), we infer

$$A_1 \leq c\|\sigma_h^k e_h^k\|_{\mathbf{L}^2}\|e_h^{k+1}\|_{\mathbf{H}^1},$$

where we estimated the second term after integrating it by parts, which is possible given the smoothness of w_h^{k+1} and e_h^{k+1}. Using (2.8) we obtain

$$A_2 \leq c\|\rho_h^{k+1} - \rho^{k+1}\|_{H^1}\|e_h^{k+1}\|_{\mathbf{H}^1},$$

where, again, we integrated by parts the second term. Finally, given the smoothness of ρ^{k+1}, an estimate of A_3 is obtained by proceeding as in the constant density case, see e.g. [13, 22]:

$$A_3 \leq c(\tau + h^{l+1})\|e_h^{k+1}\|_{\mathbf{H}^1}.$$

The estimate (4.12) is obtained by combining the results above. □

4.2. **Error Estimates.** As stated in Remark 3.2, the stability of the algorithm that we are analyzing only marginally depends on the method which is used to approximate the density; the only assumption we make to achieve stability is that the algorithm that solves the mass conservation equation satisfies (3.4). Of course (3.4) is not sufficient to obtain error estimates. Performing the full error analysis would require to analyze the nonlinear coupling between the mass conservation equation and the momentum conservation equation. This would require to be specific on the type of approximation which is used to compute the approximate density field and would probably lead to lengthy technicalities of little interest. We are not going to do the full convergence analysis to avoid technicalities and to remain as general as possible on the way the mass conservation equation is approximated. We assume instead that, in some way, we are capable of computing an approximate density sequence $\{\rho_h^k\} \subset W_h$ from the knowledge of the approximated velocity sequence $\{u_h^k\} \subset \mathbf{X}_h$. To be more specific we assume that the following holds:

$$\|(\rho - \rho_h)_\tau\|^2_{\ell^\infty(H^1)} + \left\|\left(\rho_t - \frac{\delta\rho_h}{\tau}\right)_\tau\right\|^2_{\ell^\infty(L^2)} \leq c(\lambda)(\tau + h^{l+1})^2 + \lambda\|e_h^{k+1}\|^2_{\mathbf{H}^1} + c(\lambda)\|\sigma_h^k e_h^k\|^2_{\mathbf{L}^2}, \tag{4.13}$$

where $\lambda \geq 0$ can be chosen as small as necessary. Given this assumption, the residual term $\mathcal{R}(e_h^{k+1})$ simplifies as follows:

Corollary 4.1. *Assume that* (4.13) *holds. Then, the following estimate holds under the regularity assumptions of Lemma 4.1:*

$$2\tau|\mathcal{R}^{k+1}(e_h^{k+1})| \leq c\tau(\tau + h^{l+1})^2 + \mu\tau\|e_h^{k+1}\|^2_{\mathbf{H}^1} + c\tau\|\sigma^k e_h^k\|^2_{\mathbf{L}^2}. \tag{4.14}$$

Proof. Use (4.12) where all the terms that involve differences of ρ_h and ρ can be majored by (4.13). The parameter λ is chosen so that $\lambda = \epsilon\mu$, where ϵ is chosen small enough. □

We now consider the non-incremental and the incremental versions the algorithm separately.

4.2.1. *Non-Incremental Scheme.* The non-incremental version of the method is obtained by setting $\gamma = 0$. Under assumption (4.13), the main error estimate for this algorithm is the following:

Theorem 4.1. *Assume that the solution to* (1.1)–(1.2) *satisfies* (4.2)*, and that* (3.4) *hold for all* $0 \leq k \leq K$*. Let* $(u_h)_\tau$ *be the solution of* (3.7)–(3.8) *with* $\gamma = 0$ *and assume that* (3.1) *and* (4.13) *hold. Then*

$$\|u_\tau - (u_h)_\tau\|_{\ell^\infty(\mathbf{L}^2)} \leq c\left(h^{l+1} + \tau^{1/2}\right), \qquad \|u_\tau - (u_h)_\tau\|_{\ell^2(\mathbf{H}^1)} \leq c\left(h^l + \tau^{1/2}\right). \tag{4.15}$$

Conjecture 4.1. *We expect that further regularity assumptions combined with a standard duality argument, e.g. multiplying the error equation by Se_h^{k+1}, where S is the solution operator to the time-independent Stokes problem, should allow us to conclude that the following estimate holds in addition to* (4.15)*:*

$$\|u_\tau - (u_h)_\tau\|_{\ell^2(\mathbf{L}^2)} \leq c\left(h^{l+1} + \tau\right).$$

The reader is referred to [20, 13] *for more details.*

Remark 4.1. The error estimate (4.15) shows that, at least under assumption (4.13), the non-incremental fractional time-stepping technique for variable density fluid flows performs as well as the analogous non-incremental pressure-correction scheme for constant density flows (cf. [15]).

Proof. (THEOREM 4.1) In this case $p_h^\sharp = p_h^k$ and $\phi_h^\flat = p_h^{k+1}$. Setting $r_h := 2\tau^2\epsilon_h^k/\chi$ in (4.9) we obtain

$$\frac{\tau^2}{\chi}\left[\left\|\nabla\epsilon_h^{k+1}\right\|_{\mathbf{L}^2}^2 + \left\|\nabla\epsilon_h^{k}\right\|_{\mathbf{L}^2}^2 - \left\|\nabla\delta\epsilon_h^{k+1}\right\|_{\mathbf{L}^2}^2\right] - 2\tau\left\langle\nabla\epsilon_h^k, e_h^{k+1}\right\rangle = \frac{2\tau^2}{\chi}\left\langle\nabla q_h^{k+1}, \nabla\epsilon_h^k\right\rangle. \tag{4.16}$$

Next, apply δ to (4.9) and set $r_h := \tau\delta\epsilon_h^{k+1}$. The Cauchy-Schwarz inequality implies

$$\tau^2\left\|\nabla\delta\epsilon_h^{k+1}\right\|_{\mathbf{L}^2}^2 \leq \left\|\chi\delta e_h^{k+1} + \tau\nabla\delta q_h^{k+1}\right\|_{\mathbf{L}^2}^2 = \chi^2\left\|\delta e_h^{k+1}\right\|_{\mathbf{L}^2}^2 + \tau^2\left\|\nabla\delta q_h^{k+1}\right\|_{\mathbf{L}^2}^2 + 2\chi\tau\left\langle\nabla\delta q_h^{k+1}, \delta e_h^{k+1}\right\rangle,$$

which, by (3.4), implies

$$\frac{\tau^2}{\chi}\left\|\nabla\delta\epsilon_h^{k+1}\right\|_{\mathbf{L}^2}^2 \leq \left\|\sigma_h^k\delta e_h^{k+1}\right\|_{\mathbf{L}^2}^2 + \frac{\tau^2}{\chi}\left\|\nabla\delta q_h^{k+1}\right\|_{\mathbf{L}^2}^2 + 2\tau\left\langle\nabla\delta q_h^{k+1}, \delta e_h^{k+1}\right\rangle. \tag{4.17}$$

Adding up (4.12), (4.16) and (4.17) and using Corollary 4.1, we obtain,

$$\begin{aligned}\|\sigma_h^{k+1}e_h^{k+1}\|_{\mathbf{L}^2}^2 + \mu\tau\|e_h^{k+1}\|_{\mathbf{H}^1}^2 + \frac{\tau^2}{\chi}\left[\|\nabla\epsilon_h^{k+1}\|_{\mathbf{L}^2}^2 + \|\nabla\epsilon_h^k\|_{\mathbf{L}^2}^2\right] &\leq c\tau(\tau + h^{l+1})^2 + (1+c\tau)\|\sigma_h^k e_h^k\|_{\mathbf{L}^2}^2 \\ &\quad - 2\tau\left\langle\nabla\delta q_h^{k+1}, e_h^k\right\rangle + \frac{\tau^2}{\chi}\|\delta q_h^{k+1}\|^2 + \frac{2\tau^2}{\chi}\left\langle\nabla q_h^{k+1}, \nabla\epsilon_h^k\right\rangle.\end{aligned}$$

We estimate the last three terms in the right-hand side separately. Integrating by parts and using (2.8), the first one can be estimated as follows:

$$-2\tau\left\langle\nabla\delta q_h^{k+1}, e_h^k\right\rangle \leq 2\tau\|\delta q_h^{k+1}\|_{L^2}\|e_h^k\|_{\mathbf{H}^1} \leq c\tau^3 + \frac{\mu\tau}{2}\|e_h^k\|_{\mathbf{H}^1}^2.$$

Similarly, the second term is estimated as follows:

$$\frac{\tau^2}{\chi}\|\nabla\delta q_h^{k+1}\|_{\mathbf{L}^2}^2 \leq c\tau^3.$$

For the last term, using again (2.8) we obtain

$$\frac{2\tau^2}{\chi}\left\langle\nabla q_h^{k+1}, \nabla\epsilon_h^k\right\rangle \leq c\frac{\tau^2}{\chi}\|\nabla\epsilon_h^k\|_{\mathbf{L}^2} \leq c\tau^2 + \frac{\tau^2}{\chi}\|\nabla\epsilon_h^k\|_{\mathbf{L}^2}^2.$$

Notice that this term is responsible for the loss of optimality, i.e., full first-order accuracy is lost at this point.

Combining the above observations, we finally obtain

$$\|\sigma_h^{k+1}e_h^{k+1}\|_{\mathbf{L}^2}^2 + \mu\tau\|e_h^{k+1}\|_{\mathbf{H}^1}^2 \leq (1+c\tau)\|\sigma_h^k e_h^k\|_{\mathbf{L}^2}^2 + \frac{\mu\tau}{2}\|e_h^k\|_{\mathbf{H}^1}^2 + c\tau(\tau^{1/2} + h^{l+1})^2,$$

which, by the discrete Grönwall lemma implies

$$\|(\sigma_h e_h)_\tau\|_{\ell^\infty(\mathbf{L}^2)} + \|(e_h)_\tau\|_{\ell^2(\mathbf{H}^1)} \leq c(\tau^{1/2} + h^{l+1}).$$

The claimed error estimates follow from the triangle inequality, the definition

$$u^k - u_h^k = \eta^k + e_h^k,$$

and (in the case of the $\ell^\infty(\mathbf{L}^2)$-norm) assumption (3.4). Notice that it is only at this point that the interpolation error in the $\mathbf{H}^1$-norm, which is of order $\mathcal{O}(h^l)$, is introduced. This a well-known super-convergence effect induced by our particular choice for the pair (w_h, q_h), see (2.6) and [31]. □

4.2.2. *Incremental Scheme.* The incremental version of the algorithm is obtained by setting $\gamma = 1$. Under assumption (4.13), the main error estimate for this algorithm is stated as follows:

Theorem 4.2. *Assume that the solution to* (1.1)–(1.2) *satisfies* (4.2)*, and that* (3.4) *hold for all* $0 \le k \le K$*. Let* $(u_h)_\tau$ *be the solution of* (3.7)–(3.8) *with* $\gamma = 1$ *and assume that* (3.1) *and* (4.13) *hold. Then*

$$\|u_\tau - (u_h)_\tau\|_{\ell^\infty(\mathbf{L}^2)} \le c\left(\tau + h^{l+1}\right), \tag{4.18}$$

$$\|u_\tau - (u_h)_\tau\|_{\ell^2(\mathbf{H}^1)} \le c\left(\tau + h^{l}\right). \tag{4.19}$$

Remark 4.2. The error estimates from Theorem 4.2 show that, under the given assumptions on the density approximation, the incremental pressure-correction algorithm for variable density fluid flows performs as well as the analogous incremental projection-type pressure-correction scheme for constant density flows (cf. [15]).

Proof. (THEOREM 4.2) In this case $p_h^\sharp = 2p_h^k - p_h^{k-1}$ and $\phi_h^\flat = \delta p_h^{k+1}$. Setting $r_h := -2\tau^2\delta^2\epsilon_h^{k+1}/\chi$ in (4.9), we obtain

$$-\frac{\tau^2}{\chi}\left[\|\nabla\delta\epsilon_h^{k+1}\|_{\mathbf{L}^2}^2 - \|\nabla\delta\epsilon_h^{k}\|_{\mathbf{L}^2}^2 + \|\nabla\delta^2\epsilon_h^{k+1}\|_{\mathbf{L}^2}^2\right] + 2\tau\left\langle e_h^{k+1}, \nabla\delta^2\epsilon_h^{k+1}\right\rangle = -\frac{2\tau^2}{\chi}\left\langle \nabla\delta q_h^{k+1}, \nabla\delta^2\epsilon_h^{k+1}\right\rangle.$$

Setting $r_h := 2\tau^2\epsilon_h^{k+1}/\chi$ in (4.9), we obtain

$$\frac{\tau^2}{\chi}\left[\|\nabla\epsilon_h^{k+1}\|_{\mathbf{L}^2}^2 - \|\nabla\epsilon_h^{k}\|_{\mathbf{L}^2}^2 + \|\nabla\delta\epsilon_h^{k+1}\|_{\mathbf{L}^2}^2\right] = 2\tau\left\langle e_h^{k+1}, \nabla\epsilon_h^{k+1}\right\rangle + \frac{2\tau^2}{\chi}\left\langle \nabla\delta q_h^{k+1}, \nabla\epsilon_h^{k+1}\right\rangle$$

Adding these two equations we arrive at

$$\begin{aligned}\frac{\tau^2}{\chi}\left[\|\nabla\epsilon_h^{k+1}\|_{\mathbf{L}^2}^2 - \|\nabla\epsilon_h^{k}\|_{\mathbf{L}^2}^2 + \|\nabla\delta\epsilon_h^{k}\|_{\mathbf{L}^2}^2\right] - \frac{\tau^2}{\chi}\|\nabla\delta^2\epsilon_h^{k+1}\|_{\mathbf{L}^2}^2 \\ - 2\tau\left\langle e_h^{k+1}, \nabla\epsilon_h^\sharp\right\rangle = \frac{2\tau^2}{\chi}\left\langle \nabla\delta q_h^{k+1}, \nabla\epsilon_h^\sharp\right\rangle.\end{aligned} \tag{4.20}$$

Now we apply δ to (4.9) and we set $r_h := \tau\delta^2\epsilon_h^{k+1}$. The Cauchy-Schwarz inequality implies

$$\tau^2\|\nabla\delta^2\epsilon_h^{k+1}\|_{\mathbf{L}^2}^2 \le \left\|\chi\delta e_h^{k+1} + \tau\nabla\delta^2 q_h^{k+1}\right\|_{\mathbf{L}^2}^2 = \chi^2\|\delta e_h^{k+1}\|_{\mathbf{L}^2}^2 + \tau^2\|\nabla\delta^2 q_h^{k+1}\|_{\mathbf{L}^2}^2 + 2\tau\chi\left\langle \nabla\delta^2 q_h^{k+1}, \delta e_h^{k+1}\right\rangle,$$

and owing to (3.4) we infer

$$\frac{\tau^2}{\chi}\|\nabla\delta^2\epsilon_h^{k+1}\|_{\mathbf{L}^2}^2 \le \|\sigma^k\delta e_h^{k+1}\|_{\mathbf{L}^2}^2 + \frac{\tau^2}{\chi}\|\nabla\delta^2 q_h^{k+1}\|_{\mathbf{L}^2}^2 + 2\tau\left\langle \nabla\delta^2 q_h^{k+1}, \delta e_h^{k+1}\right\rangle. \tag{4.21}$$

Adding (4.11), (4.20) and (4.21), and using Corollary 4.1, we arrive at

$$\|\sigma_h^{k+1}e_h^{k+1}\|_{\mathbf{L}^2}^2 + \mu\tau\|e_h^{k+1}\|_{\mathbf{H}^1}^2 + \frac{\tau^2}{\chi}[\|\nabla\epsilon_h^{k+1}\|_{\mathbf{L}^2}^2 + \|\nabla\delta\epsilon_h^{k+1}\|_{\mathbf{L}^2}^2] \leq (1+c\tau)\|\sigma_h^k e_h^k\|_{\mathbf{L}^2}^2 + \frac{\tau^2}{\chi}\|\nabla\epsilon_h^k\|_{\mathbf{L}^2}^2$$
$$+ c\tau(\tau+h^{l+1})^2 + \frac{\tau^2}{\chi}\|\nabla\delta^2 q_h^{k+1}\|_{\mathbf{L}^2}^2 - 2\tau\left\langle\nabla\delta^2 q_h^{k+1}, e_h^k\right\rangle + \frac{2\tau^2}{\chi}\left\langle\nabla\delta q_h^{k+1}, \nabla\epsilon_h^\sharp\right\rangle.$$

Let us estimate the last three terms separately. Clearly,

$$\tau^2/\chi\|\nabla\delta^2 q_h^{k+1}\|_{\mathbf{L}^2}^2 \leq c\tau^3.$$

The second term is bounded from above as follows:

$$-2\tau\left\langle\nabla\delta^2 q_h^{k+1}, e_h^k\right\rangle \leq c\tau^3 + \frac{\mu\tau}{2}\|e_h^k\|_{\mathbf{H}^1}^2.$$

Finally, for the third term we have

$$\frac{2\tau^2}{\chi}\left\langle\nabla\delta q_h^{k+1}, \nabla\epsilon_h^\sharp\right\rangle \leq c\tau^3 + \tau^3\|\nabla\epsilon_h^k\|_{\mathbf{L}^2}^2 + \tau^3\|\nabla\delta\epsilon_h^k\|_{\mathbf{L}^2}^2.$$

We obtain the estimate (4.18)-(4.19) by finishing as in the proof of Theorem 4.1. □

5. A Second-Order Fractional Time-Stepping Method

We have established in the previous section that the incremental version of the scheme (3.4)–(3.9) is first-order accurate in time both for the $\mathbf{L}^2$- and the $\mathbf{H}^1$-norm of the velocity. However, as shown in [13], we expect that the splitting error of the algorithm is second-order since the pressure term

$$p_h^\sharp = 2p_h^k - p_h^{k-1}, \tag{5.1}$$

that appears in the approximate momentum equation is a second-order extrapolation of the pressure p_h^{k+1}. This observation is the main motivation for our introducing a variant of the incremental method using a second-order approximation of the time derivative of the velocity.

This section is organized as follows. In §5.1 we introduce the second-order algorithm and we derive some of its immediate properties. In §5.2 we prove estimates for the solutions of a three term recursion inequality which is used to prove stability of the algorithm. In §5.3 we reprove stability of a second-order projection scheme for the time-dependent Stokes equations with constant density. Although the result *per se* is not new (see [13, 15, 9]), to the best of our knowledge the technique that we use is new. The originality of the proof technique is that the so-called projected velocity is totally eliminated from the analysis. This trick enables us to easily extend the proof to the variable density case. The stability proof is reported in §5.4.

5.1. Description of the Algorithm.

Keeping the same notation as in the previous sections, the second-order variant of the algorithm is composed of the following steps:

5.1.1. *Initialization.* First, we choose a penalty parameter χ as in §3.1. Next, we define $(\rho_h^0, u_h^0, p_h^0, \phi_h^0 = 0) \in W_h\times\mathbf{X}_h\times M_h\times M_h$ to be a suitable approximation of the initial data of the problem. Then we compute an approximation of the exact solution at time $t=\tau$, say $(\rho_h^1, u_h^1, p_h^1, \phi_h^1 = p_h^1 - p_h^0) \in W_h\times\mathbf{X}_h\times M_h\times M_h$.

5.1.2. *Time-Stepping.* Given $(\rho_h^k, u_h^k, p_h^k, \phi_h^k) \in W_h\times\mathbf{X}_h\times M_h\times M_h$ for $1 \leq k \leq K-1$, we compute the next time-step approximation as follows:

5.1.3. *Density Update.* We are not specific on the way $\rho_h^{k+1} \in W_h$ is computed, but we assume that (3.4) holds and that there is a uniform constant M so that

$$\max_{0\le k\le K-1} \left\| \frac{\rho_h^{k+1}-\rho_h^k}{\tau} \right\|_{L^\infty} \le M\chi. \tag{5.2}$$

5.1.4. *Velocity Update.* Similarly to §3.2.2 we define

$$\rho_h^\star := \frac{3}{2}\rho_h^{k+1} - \frac{2}{3}\rho_h^k + \frac{1}{6}\rho_h^{k-1} = \rho_h^{k+1} + \frac{1}{6}(3\rho_h^{k+1} - 4\rho_h^k + \rho_h^{k-1}), \tag{5.3}$$

$$p_h^\sharp := p_h^k + \frac{4}{3}\phi_h^k - \frac{1}{3}\phi_h^{k-1}. \tag{5.4}$$

Then we compute $u_h^{k+1} \in \mathbf{X}_h$ so that the following holds:

$$\begin{aligned} \left\langle \frac{3\rho_h^\star u_h^{k+1} - 4\rho_h^{k+1}u_h^k + \rho_h^{k+1}u_h^{k-1}}{2\tau}, v_h \right\rangle + \left\langle \rho_h^{k+1} u_h^\star \cdot\nabla u_h^{k+1} + \frac{1}{2} u_h^{k+1} \nabla\cdot(\rho_h^{k+1}u_h^\star), v_h \right\rangle \\ + \mu\left\langle \nabla u_h^{k+1}, \nabla v_h\right\rangle + \left\langle \nabla p_h^\sharp, v_h\right\rangle = \left\langle f^{k+1}, v_h\right\rangle, \quad \forall v_h \in \mathbf{X}_h, \end{aligned} \tag{5.5}$$

where

$$u_h^\star := 2u_h^k - u_h^{k-1}, \tag{5.6}$$

is a second-order extrapolation of the velocity.

5.1.5. *Penalty.* We compute the pressure correction $\phi_h^{k+1} \in M_h$ so that the following holds:

$$\left\langle \nabla\phi_h^{k+1}, \nabla r_h\right\rangle = \frac{3\chi}{2\tau}\left\langle u_h^{k+1}, \nabla r_h\right\rangle, \quad \forall r_h \in M_h. \tag{5.7}$$

5.1.6. *Pressure Update.* Finally, the pressure is updated by setting

$$p_h^{k+1} = p_h^k + \phi_h^{k+1}. \tag{5.8}$$

Remark 5.1. The quantities $(\rho_h^1, u_h^1, p_h^1, \phi_h^1)$ can be computed by using one step of the incremental first-order scheme described in Section 3.

Remark 5.2. The term $\left\langle \frac{1}{2}\nabla\cdot(\rho_h^{k+1}u_h^\star)u_h^{k+1}, v_h\right\rangle$ has been added to the equation to obtain unconditional stability with respect to the advection term. As in the proof of Lemma 4.1 we are going to use the following identity:

$$\left\langle \rho_h^{k+1}u_h^\star\cdot\nabla u_h^{k+1} + \frac{1}{2}\nabla\cdot(\rho_h^{k+1}u_h^\star)u_h^{k+1}, u_h^{k+1}\right\rangle = \int_\Omega \rho_h^{k+1}u_h^\star\cdot\nabla u_h^{k+1}\cdot u_h^{k+1} + \frac{1}{2}\int_\Omega \nabla\cdot(\rho_h^{k+1}u_h^\star)|u_h^{k+1}|^2 = 0.$$

Remark 5.3. The term

$$\frac{3\rho_h^\star u_h^{k+1} - 4\rho_h^{k+1}u_h^k + \rho_h^{k+1}u_h^{k-1}}{2\tau} + \frac{1}{2}\nabla\cdot(\rho_h^{k+1}u_h^\star)u_h^{k+1},$$

is a second-order approximation of $[\rho_h u_{h,t}](t^{k+1})$. Indeed, if the involved functions are smooth enough in time, we infer from the definition of $\rho_h^\star$ that

$$\frac{3\rho_h^\star u_h^{k+1} - 4\rho_h^{k+1} u_h^k + \rho_h^{k+1} u_h^{k-1}}{2\tau} + \frac{1}{2}\nabla\cdot(\rho_h^{k+1} u_h^\star) u_h^{k+1} =$$
$$\frac{\rho_h^{k+1}}{2\tau}(3u_h^{k+1} - 4u_h^k + u_h^{k-1}) + \frac{1}{2}\left(\frac{3\rho_h^{k+1} - 4\rho_h^k + \rho_h^{k-1}}{2\tau} + \nabla\cdot(\rho_h^{k+1} u_h^\star)\right) u_h^{k+1} =$$
$$[\rho_h u_{h,t}]^{k+1} + \frac{1}{2}\left[\rho_{h,t} + \nabla\cdot(\rho_h u_h)\right]^{k+1} u_h^{k+1} + \mathcal{O}(\tau^2) = [\rho_h u_{h,t}]^{k+1} + \mathcal{O}(\tau^2),$$

which proves the claim.

5.2. **Three Term Recursion Inequalities.** We prove in this section preliminary results regarding three term recursion inequalities. These results will be needed to prove stability of the algorithm (5.3)–(5.8).

Proposition 5.1. *Assume that the characteristic polynomial of the three term recursion equation*

$$Ax^{k+1} + Bx^k + Cx^{k-1} = g^{k+1}, \qquad k \geq 2 \tag{5.9}$$

has two (not necessarily distinct) nonzero real roots r_1 and r_2. Then, the generic solution to this equation has the form

$$x^\nu = c_1 r_1^\nu + c_2 r_2^\nu + \frac{1}{A}\sum_{l=2}^{\nu} r_1^{\nu-l} \sum_{s=2}^{l} r_2^{l-s} g^s, \quad c_1, c_2 \in \mathbb{R}.$$

Proof. It is sufficient to show that

$$\bar{x}^\nu = \frac{1}{A}\sum_{l=2}^{\nu} r_1^{\nu-l} \sum_{s=2}^{l} r_2^{l-s} g^s, \quad \nu \geq 2,$$

with $\bar{x}^1 = \bar{x}^0 = 0$ is a particular solution of (5.9).

Let $n \geq 1$. Multiply (5.9) by r_2^{2n-k-2} and add all the results for $k = 1, \ldots, n$. Setting $x^1 = x^0 = 0$, we obtain

$$Ar_2^{n-2} x^{n+1} + r_2^{n-2}(Ar_2 + B)x^n + \sum_{k=2}^{n-1}\left[(Ar_2^{2n-k-1} + Br_2^{2n-k-2} + Cr_2^{2n-k-3})x^k\right] = \sum_{s=2}^{n+1} r_2^{2n-s-1} g^s,$$

which, since r_2 is a root of the characteristic polynomial, implies

$$Ax^{n+1} + (Ar_2 + B)x^n = \sum_{s=2}^{n+1} r_2^{n+1-s} g^s, \quad n \geq 2. \tag{5.10}$$

Let $\nu \geq 1$. Multiply (5.10) by $r_1^{\nu-n}$ and add all the results for $n = 1, \ldots, \nu$. We obtain

$$Ax^{\nu+1} + \sum_{l=2}^{\nu}\left[r_1^{\nu-l}(A(r_1 + r_2) + B)x^l\right] = \sum_{l=2}^{\nu+1} r_1^{\nu+1-l} \sum_{s=2}^{l} r_2^{l-s} g^s, \quad \nu \geq 1.$$

Since r_1, r_2 are roots of the characteristic polynomial of the recursion equation, we have $B = -(r_1 + r_2)A$, which implies

$$x^{\nu+1} = \frac{1}{A}\sum_{l=2}^{\nu+1} r_1^{\nu+1-l} \sum_{s=2}^{l} r_2^{l-s} g^s, \quad \nu \geq 1.$$

Hence, $\bar{x}^\nu$ is a particular solution of (5.9). ☐

Proposition 5.2. *Assume that the coefficients of the three term recursion inequality*

$$Ay^{k+1} + By^k + Cy^{k-1} \le g^{k+1}, \quad k \ge 1, \tag{5.11}$$

satisfy

$$A > 0, \quad C \ge 0, \quad A + B + C \le 0.$$

Let $\{y^k\}_{k\ge0}$ *be a solution to* (5.11) *with initial data* y^0 *and* y^1*. If* $\{x^k\}_{k\ge0}$ *solves* (5.9) *with initial data* $x^0 = y^0$ *and* $x^1 = y^1$*, then the following estimate holds*

$$y^\nu \le x^\nu, \quad \forall \nu \ge 0.$$

Proof. This is a comparison argument *à la* Grönwall. Let $\{z^k\}_{k\ge0}$ be the sequence defined by $z^\nu = y^\nu - x^\nu$. Let us prove by induction that $z^k \le z^{k-1}$, for all $k \ge 1$. The claim holds true for $k = 1$ since $0 = z^1 \le z^0 = 0$. Assume now that $z^\nu \le z^{\nu-1}$ for all $1 \le \nu \le k$. The definition of $\{x^k\}_{k\ge0}$ implies

$$Az^{k+1} + Bz^k + Cz^{k-1} \le 0, \quad \forall k \ge 1.$$

Hence

$$Az^{k+1} \le Az^k - (A + B + C)z^k + C(z^k - z^{k-1}) \le Az^k,$$

which proves the claim. ☐

The following corollary is a specialization of the two previous results which will be needed in the sequel.

Corollary 5.1. *The three term recursion equation*

$$3x^{k+1} - 4x^k + x^{k-1} = g^{k+1}, \quad k \ge 1, \tag{5.12}$$

has the following general solution

$$x^\nu = c_1 + \frac{c_2}{3^\nu} + \sum_{l=2}^{\nu} \frac{1}{3^{\nu+1-l}} \sum_{s=2}^{l} g^s, \qquad c_1 \in \mathbb{R},\ c_2 \in \mathbb{R}.$$

Let $\{y^k\}_{k\ge0}$ *be the solution to the three term recursion inequality*

$$3y^{k+1} - 4y^k + y^{k-1} \le g^{k+1}, \quad k \ge 1,$$

with initial data y^0 *and* y^1*. If* $\{x^k\}_{k\ge0}$ *is the solution to* (5.12) *with initial data* $x^0 = y^0$ *and* $x^1 = y^1$*, then the following estimate holds*

$$y^\nu \le x^\nu, \quad \forall \nu \ge 0.$$

Proof. To obtain the generic solution, it is sufficient to notice that the roots of the characteristic polynomial of the equation are $r_2 = 1$ and $r_1 = 1/3$. To obtain the estimate it is sufficient to notice that $A = 3 > 0$, $C = 1 > 0$ and $A + B + C = 3 - 4 + 1 = 0 \le 0$. ☐

5.3. **Stability of the Incremental Pressure-Correction Algorithm in Standard Form.** The objective of this section is to prove stability estimates for the incremental pressure-correction algorithm in standard form for the approximation of the time-dependent Stokes equation with constant density. This result is not new but the technique that we use to prove these estimates gives insight on the way to proceed when the density is variable. The main novelty is that the proof does not use the projected velocity. To the best of the authors' knowledge this proof technique has never been used before.

We restrict ourselves for the time being to the time-dependent Stokes equations:

$$\begin{cases} u_t - \Delta u + \nabla p = f, \quad u|_{t=0} = u_0, \quad u|_\Gamma = 0, \\ \nabla{\cdot} u = 0. \end{cases}$$

Without going into the details (for which we refer the reader to [9, 13, 15]) we now describe the incremental pressure-correction projection in standard form. After proper initialization, given $(u_h^k, p_h^k, \phi_h^k) \in \mathbf{X}_h \times M_h \times M_h$ the next iterate is computed in three steps:
(i) Find $u_h^{k+1} \in \mathbf{X}_h$ so that

$$\left\langle \frac{3u_h^{k+1} - 4u_h^k + u_h^{k-1}}{2\tau}, v_h \right\rangle + \left\langle \nabla u_h^{k+1}, \nabla v_h \right\rangle + \left\langle \nabla p_h^\sharp, v_h \right\rangle = \left\langle f^{k+1}, v_h \right\rangle, \tag{5.13}$$

where $p_h^\sharp$ is defined in (5.4).
(ii) Find $\phi_h^{k+1} \in M_h$ so that

$$\left\langle \nabla \phi_h^{k+1}, \nabla r_h \right\rangle = \frac{3}{2\tau} \left\langle u_h^{k+1}, \nabla r_h \right\rangle \tag{5.14}$$

(iii) Finally, update the pressure by

$$p_h^{k+1} = p_h^k + \phi_h^{k+1}. \tag{5.15}$$

We now prove that the above algorithm is stable. To avoid irrelevant technicalities we assume that $f \equiv 0$.

Theorem 5.1. *The solution* $\{(u_h^k, p_h^k)\}_{k\geq 0} \subset \mathbf{X}_h \times M_h$ *to* (5.13)-(5.14)-(5.15) *satisfies the following estimate:*

$$\begin{aligned} \|u_h^k\|_{\mathbf{L}^2}^2 + \tau^2 \|\nabla p_h^k\|_{\mathbf{L}^2}^2 + \tau^2 \|\nabla \delta p_h^{k-1}\|_{\mathbf{L}^2}^2 + \sum_{l=2}^{k} \left[\tau \|u_h^l\|_{\mathbf{H}^1}^2 + \tau^2 \|\nabla \delta p_h^{l-1}\|_{\mathbf{L}^2}^2 \right] \\ \leq c \left(\|u_h^0\|_{\mathbf{L}^2}^2 + \|u_h^1\|_{\mathbf{L}^2}^2 + \tau^2 \|\nabla p_h^0\|_{\mathbf{L}^2}^2 + \tau^2 \|\nabla p_h^1\|_{\mathbf{L}^2}^2 \right), \quad \forall k \geq 2. \end{aligned}$$

Proof. We proceed in two steps:
(i) *Initialization:* We consider the steps $k = 1, 2$ separately as they involve the initial quantities. Let us begin by noticing that the definition of $p_h^\sharp$ involves only terms from the previous time steps. For $k = 1$ or 2 we set $v_h := 4\tau u_h^{k+1}$ in (5.13). Then using the identity

$$2a^{k+1}\left(3a^{k+1} - 4a^k + a^{k-1}\right) = \left|a^{k+1}\right|^2 + \left|2a^{k+1} - a^k\right|^2 + \left|\delta^2 a^{k+1}\right|^2 - \left|a^k\right|^2 - \left|2a^k - a^{k-1}\right|^2,$$

and the Cauchy-Schwarz inequality we obtain

$$\frac{1}{2}\|u_h^{k+1}\|_{\mathbf{L}^2}^2 + \|2u_h^{k+1} - u^k\|_{\mathbf{L}^2}^2 + 4\tau \|u_h^{k+1}\|_{\mathbf{H}^1}^2 \leq \|u_h^k\|_{\mathbf{L}^2}^2 + \|2u_h^k - u_h^{k-1}\|_{\mathbf{L}^2}^2 + 8\tau^2 \|\nabla p_h^\sharp\|_{\mathbf{L}^2}^2,$$

which implies

$$\|u_h^{k+1}\|_{\mathbf{L}^2}^2 + \tau\|u_h^{k+1}\|_{\mathbf{H}^1}^2 \le c\left(\|u_h^0\|_{\mathbf{L}^2}^2 + \|u_h^1\|_{\mathbf{L}^2}^2 + \tau^2\|\nabla p_h^0\|_{\mathbf{L}^2}^2 + \tau^2\|\nabla p_h^1\|_{\mathbf{L}^2}^2\right),$$

for $k = 1$ or 2. The estimate on the pressure is obtained using equations (5.14)–(5.15) as follows:

$$\frac{4\tau^2}{9}\|\nabla\delta p_h^{k+1}\|_{\mathbf{L}^2}^2 \le \|u_h^{k+1}\|_{\mathbf{L}^2}^2.$$

The triangle inequality and the estimates obtained above imply the claimed estimate for the first two steps $k = 1, 2$.

(ii) *General Step:* For $k \ge 3$ notice that, by (5.15)

$$p_h^\sharp = \frac{7p_h^k - 5p_h^{k-1} + p_h^{k-2}}{3} = \frac{3p_h^{k+1} - 3\delta^2 p_h^{k+1} + \delta^2 p_h^k}{3}.$$

Setting $v_h := 4\tau u_h^{k+1}$ in (5.13), and using the identity

$$2a^{k+1}\left(3a^{k+1} - 4a^k + a^{k-1}\right) = 3\left|a^{k+1}\right|^2 - 4\left|a^k\right|^2 + \left|a^{k-1}\right|^2 + 2\left|\delta a^{k+1}\right|^2 - 2\left|\delta a^k\right|^2 + \left|\delta^2 a^{k+1}\right|^2,$$

we obtain

$$\begin{aligned}3\|u_h^{k+1}\|_{\mathbf{L}^2}^2 - 4\|u_h^k\|_{\mathbf{L}^2}^2 + \|u_h^{k-1}\|_{\mathbf{L}^2}^2 + 2\|\delta u_h^{k+1}\|_{\mathbf{L}^2}^2 - 2\|\delta u_h^k\|_{\mathbf{L}^2}^2 + \|\delta^2 u_h^{k+1}\|_{\mathbf{L}^2}^2 + 4\tau\|u_h^{k+1}\|_{\mathbf{H}^1}^2 \\ + 4\tau\left\langle\nabla p_h^{k+1}, u_h^{k+1}\right\rangle - 4\tau\left\langle\nabla\delta^2 p_h^{k+1}, u_h^{k+1}\right\rangle + \frac{4\tau}{3}\left\langle\nabla\delta^2 p_h^k, u_h^{k+1}\right\rangle = 0.\end{aligned}$$

From the projection equation (5.14) and the pressure update equation (5.15) we deduce that

$$\left\langle\nabla r_h, u_h^{k+1}\right\rangle = \frac{2\tau}{3}\left\langle\nabla r_h, \nabla\delta p_h^{k+1}\right\rangle, \quad r_h \in M_h.$$

Using this property together with the identity $2a(a-b) = a^2 - b^2 + (a-b)^2$ we infer

$$\begin{aligned}3\|u_h^{k+1}\|_{\mathbf{L}^2}^2 - 4\|u_h^k\|_{\mathbf{L}^2}^2 + \|u_h^{k-1}\|_{\mathbf{L}^2}^2 + 2\|\delta u_h^{k+1}\|_{\mathbf{L}^2}^2 - 2\|\delta u_h^k\|_{\mathbf{L}^2}^2 + \|\delta^2 u_h^{k+1}\|_{\mathbf{L}^2}^2 + 4\tau\|u_h^{k+1}\|_{\mathbf{H}^1}^2 \\ + \frac{4\tau^2}{3}\left[\|\nabla p_h^{k+1}\|_{\mathbf{L}^2}^2 - \|\nabla p_h^k\|_{\mathbf{L}^2}^2 + \|\nabla\delta p_h^k\|_{\mathbf{L}^2}^2 - \|\nabla\delta^2 p_h^{k+1}\|_{\mathbf{L}^2}^2\right] + \frac{8\tau^2}{9}\left\langle\nabla\delta^2 p_h^k, \nabla\delta p_h^{k+1}\right\rangle = 0.\end{aligned}$$

Now we use the following identity:

$$\|\delta u_h\|_{\mathbf{L}^2}^2 = \|\delta u_h - \frac{2\tau}{3}\nabla\delta^2 p_h\|_{\mathbf{L}^2}^2 + \frac{4\tau^2}{9}\|\nabla\delta^2 p_h\|_{\mathbf{L}^2}^2$$

which we apply at time steps t^{k+1} and t^k (note that it is critical to have $k \ge 3$ here) and we obtain

$$\begin{aligned}&3\|u_h^{k+1}\|_{\mathbf{L}^2}^2 - 4\|u_h^k\|_{\mathbf{L}^2}^2 + \|u_h^{k-1}\|_{\mathbf{L}^2}^2 + \|\delta^2 u_h^{k+1}\|_{\mathbf{L}^2}^2 + 4\tau\|u_h^{k+1}\|_{\mathbf{H}^1}^2 \\ &+ 2\|\delta u_h^{k+1} - \frac{2\tau}{3}\nabla\delta^2 p_h^{k+1}\|_{\mathbf{L}^2}^2 - 2\|\delta u_h^k - \frac{2\tau}{3}\nabla\delta^2 p_h^k\|_{\mathbf{L}^2}^2 + \frac{4\tau^2}{3}\left[\|\nabla p_h^{k+1}\|_{\mathbf{L}^2}^2 - \|\nabla p_h^k\|_{\mathbf{L}^2}^2 + \|\nabla\delta p_h^k\|_{\mathbf{L}^2}^2\right] \\ &\qquad - \frac{4\tau^2}{9}\|\nabla\delta^2 p_h^{k+1}\|_{\mathbf{L}^2}^2 - \frac{8\tau^2}{9}\|\nabla\delta^2 p_h^k\|_{\mathbf{L}^2}^2 + \frac{8\tau^2}{9}\left\langle\nabla\delta^2 p_h^k, \nabla\delta p_h^{k+1}\right\rangle = 0.\end{aligned}$$

We observe from this inequality that we need to control the last three terms. We rewrite these as follows:

$$\begin{aligned}&-\frac{4\tau^2}{9}\|\nabla\delta^2 p_h^{k+1}\|_{\mathbf{L}^2}^2 - \frac{8\tau^2}{9}\|\nabla\delta^2 p_h^k\|_{\mathbf{L}^2}^2 + \frac{8\tau^2}{9}\left\langle\nabla\delta^2 p_h^k, \nabla\delta p_h^{k+1}\right\rangle = \\ &\qquad -\frac{4\tau^2}{9}\|\nabla\delta^3 p_h^{k+1}\|_{\mathbf{L}^2}^2 - \frac{4\tau^2}{9}\left\langle\nabla\delta^2 p_h^k, \nabla\left(\delta^2 p_h^k + 2\delta^2 p_h^{k+1} - 2\delta p_h^{k+1}\right)\right\rangle.\end{aligned}$$

Applying δ^2 to (5.14) and using the Cauchy-Schwarz inequality we obtain, for $k \geq 3$

$$\frac{4\tau^2}{9}\|\nabla\delta^3 p_h^{k+1}\|^2_{\mathbf{L}^2} \leq \|\delta^2 u_h^{k+1}\|^2_{\mathbf{L}^2}.$$

Observing that $\delta^2 p_h^k + 2\delta^2 p_h^{k+1} - 2\delta p_h^{k+1} = -\delta p_h^k - \delta p_h^{k-1}$ and using the inequality above, we obtain the following bound:

$$\begin{aligned}-\frac{4\tau^2}{9}\|\delta^2 p_h^{k+1}\|^2 - \frac{8\tau^2}{9}\|\delta^2 p_h^k\|^2 + \frac{8\tau^2}{9}\left\langle\nabla\delta^2 p_h^k, \nabla\delta p_h^{k+1}\right\rangle \geq \\ -\|\delta^2 u_h^{k+1}\|^2_{\mathbf{L}^2} + \frac{4\tau^2}{9}\left[\|\delta p_h^k\|^2 - \|\delta p_h^{k-1}\|^2\right],\end{aligned}$$

from which we finally deduce the following energy estimate:

$$\begin{aligned}3\|u_h^{k+1}\|^2_{\mathbf{L}^2} - 4\|u_h^k\|^2_{\mathbf{L}} + \|u_h^{k-1}\|^2_{\mathbf{L}^2} + 4\tau\|u_h^{k+1}\|^2_{\mathbf{H}^1} \\ + 2\|\delta u_h^{k+1} - \frac{2\tau}{3}\nabla\delta^2 p_h^{k+1}\|^2_{\mathbf{L}^2} - 2\|\delta u_h^k - \frac{2\tau}{3}\nabla\delta^2 p_h^k\|^2_{\mathbf{L}^2} \\ + \frac{4\tau^2}{3}\left[\|\nabla p_h^{k+1}\|^2_{\mathbf{L}^2} - \|\nabla p_h^k\|^2_{\mathbf{L}^2} + \|\nabla\delta p_h^k\|^2_{\mathbf{L}^2}\right] + \frac{4\tau^2}{9}\left[\|\nabla\delta p_h^k\|^2_{\mathbf{L}^2} - \|\nabla\delta p_h^{k-1}\|^2_{\mathbf{L}^2}\right] \leq 0.\end{aligned} \tag{5.16}$$

We are now going to use the stability estimates proved in §5.2. Let us define the quantities

$$\begin{aligned}a^s &:= \|u_h^s\|^2_{\mathbf{L}^2}, \\ b^s &:= 4\tau\|u_h^s\|^2_{\mathbf{H}^1} + \frac{4\tau^2}{3}\|\nabla\delta p_h^{s-1}\|^2_{\mathbf{L}^2}, \\ d^s &:= 2\|\delta u_h^s - \frac{2\tau}{3}\nabla\delta^2 p_h^s\|^2_{\mathbf{L}^2} + \frac{4\tau^2}{3}\|\nabla p_h^s\|^2_{\mathbf{L}^2} + \frac{4\tau^2}{9}\|\nabla\delta p_h^{s-1}\|^2_{\mathbf{L}^2}.\end{aligned}$$

Then (5.16) can be rewritten as

$$3a^{k+1} - 4a^k + a^{k-1} \leq -\left(b^{k+1} + d^{k+1} - d^k\right), \quad k \geq 3$$

Setting $g^{k+1} := -\left(b^{k+1} + d^{k+1} - d^k\right)$ this three-term recursion inequality satisfies the hypotheses of Corollary 5.1 for $k \geq 3$. Hence

$$a^\nu \leq c\left(a^1 + a^2\right) - \sum_{l=3}^{\nu}\frac{1}{3^{\nu+1-l}}\sum_{s=3}^{l}\left(b^s + d^s - d^{s-1}\right), \quad \nu \geq 3$$

or

$$a^\nu + \sum_{l=3}^{\nu}\frac{1}{3^{\nu+1-l}}d^l + \sum_{l=3}^{\nu}\frac{1}{3^{\nu+1-l}}\sum_{s=3}^{l}b^s \leq c\left(a^1 + a^2 + d^2\right), \quad \nu \geq 3.$$

Dropping some positive terms in the left-hand side we deduce

$$a^\nu + \frac{1}{3}d^\nu + \frac{1}{3}\sum_{s=2}^{\nu}b^s \leq c\left(a^1 + a^2 + d^2\right).$$

Given the bounds obtained in the initialization step, this inequality implies the claimed result. □

5.4. **Stability of the Second-Order Fractional Time-Stepping Scheme For Variable Density Flows.** We now establish stability for the algorithm (5.5)-(5.7)-(5.8). Again, to avoid irrelevant technicalities, assume that $f \equiv 0$. The main result of this section is the following:

Theorem 5.2. *Assume that the sequence of approximate densities* $\{\rho_h^k\}_{k\geq 0} \subset W_h$ *satisfies* (3.4) *and* (5.2). *Then, for* τ *small enough, the sequence* $\{(u_h^k, p_h^k)\}_{k\geq 0} \subset \mathbf{X}_h \times M_h$ *obtained by the algorithm* (5.5)-(5.7)-(5.8) *satisfies the following estimate:*

$$\begin{aligned}(5.17)\quad \|\sigma_h^k u_h^k\|_{\mathbf{L}^2}^2 + \mu\tau\|u_h^k\|_{\mathbf{H}^1}^2 + \frac{\tau^2}{\chi}\|\nabla p_h^k\|_{\mathbf{L}^2}^2 + \frac{\tau^2}{\chi}\|\nabla \delta p_h^{k-1}\|^2 \leq \\ \mathcal{K}(1+e^{cT})\left(\|\sigma_h^0 u_h^0\|_{\mathbf{L}^2}^2 + \|\sigma_h^1 u_h^1\|_{\mathbf{L}^2}^2 + \|\nabla p_h^0\|_{\mathbf{L}^2}^2 + \|\nabla p_h^1\|_{\mathbf{L}^2}^2\right), \quad \forall k \geq 2,\end{aligned}$$

for some constants c *and* $\mathcal{K}$.

Proof. Note first that, as already mentioned in Remark 5.3, the time derivative can be re-written as follows:

$$\frac{3\rho_h^\star u_h^{k+1} - 4\rho_h^{k+1}u_h^k + \rho_h^{k+1}u_h^{k-1}}{2\tau} = \rho^{k+1}\frac{3u_h^{k+1} - 4u_h^k + u_h^{k-1}}{2\tau} + \frac{1}{2}u_h^{k+1}\frac{3\rho^{k+1} - 4\rho^k + \rho^{k-1}}{2\tau},$$

which is an approximation of $\rho\partial_t u + \frac{1}{2}u\partial_t\rho$. Once tested with u, the expression $(\rho\partial_t u + \frac{1}{2}u\partial_t\rho)u$ gives $\partial_t(\frac{1}{2}\rho u^2)$, and after integration over Ω and over the time interval $(0,T)$ this yields kinetic energy conservation. We have been able to reproduce this argument at the discrete level for the first-order time stepping described in § 4, see (4.11). Unfortunately, we have not yet figured out how to repeat this argument with BDF2. We are going to content ourselves with a sub-optimal stability analysis which will yield the growth constant $(1+e^{cT})$ in (5.17).

Using Assumption (5.2), we have the following estimate

$$\begin{aligned}\left\langle\left(3\rho_h^{k+1} - 4\rho_h^k + \rho_h^{k-1}\right)u_h^{k+1}, u_h^{k+1}\right\rangle &= 3\int_\Omega \left(\rho_h^{k+1} - \rho_h^k\right)|u_h^{k+1}|^2 - \int_\Omega \left(\rho_h^k - \rho_h^{k-1}\right)|u_h^{k+1}|^2 \\ &\geq -\left(3\left\|\frac{\rho_h^{k+1} - \rho_h^k}{\chi}\right\|_{L^\infty} + \left\|\frac{\rho_h^k - \rho_h^{k-1}}{\chi}\right\|_{L^\infty}\right)\|\sigma_h^{k+1}u_h^{k+1}\|_{\mathbf{L}^2}^2 \\ &\geq -4M\tau\|\sigma_h^{k+1}u_h^{k+1}\|_{\mathbf{L}^2}^2.\end{aligned}$$

A similar treatment gives

$$\begin{aligned}2\left\langle\rho_h^{k+1}\left(3u_h^{k+1} - 4u_h^k + u_h^{k-1}\right), u_h^{k+1}\right\rangle &\geq 3\|\sigma_h^{k+1}u_h^{k+1}\|_{\mathbf{L}^2}^2 - (4+8M\tau)\|\sigma_h^k u_h^k\|_{\mathbf{L}^2}^2 \\ &+ (1-6M\tau)\|\sigma_h^{k-1}u_h^{k-1}\|_{\mathbf{L}^2}^2 + 2\|\sigma_h^{k+1}\delta u_h^{k+1}\|_{\mathbf{L}^2}^2 - 2\|\sigma_h^k\delta u_h^k\|_{\mathbf{L}^2}^2 + \|\sigma_h^{k+1}\delta^2 u_h^{k+1}\|_{\mathbf{L}^2}^2.\end{aligned}$$

Combining the above two inequalities gives

$$\begin{aligned}(5.18)\quad 2\left\langle 3\rho_h^\star u_h^{k+1} - 4\rho_h^{k+1}u_h^k + \rho_h^{k+1}u_h^{k-1}, u_h^{k+1}\right\rangle &\geq (3-4M\tau)\|\sigma_h^{k+1}u_h^{k+1}\|_{\mathbf{L}^2}^2 \\ &- (4+8M\tau)\|\sigma_h^k u_h^k\|_{\mathbf{L}^2}^2 + (1-6M\tau)\|\sigma_h^{k-1}u_h^{k-1}\|_{\mathbf{L}^2}^2 \\ &+ 2\|\sigma_h^{k+1}\delta u_h^{k+1}\|_{\mathbf{L}^2}^2 - 2\|\sigma_h^k\delta u_h^k\|_{\mathbf{L}^2}^2 + \|\sigma_h^{k+1}\delta^2 u_h^{k+1}\|_{\mathbf{L}^2}^2.\end{aligned}$$

This estimate will be used repeatedly.

Now we proceed in two steps, as in the proof of Theorem 5.1: First we investigate the time steps $k = 1, 2$, then we investigate the cases $k \geq 3$.

(i) *Initialization:* Let $k \in \{1,2\}$ and set $v := 4\tau u_h^{k+1}$ in (5.5). Using (5.18) and the Cauchy-Schwarz inequality we get,

$$(3-4M\tau)\|\sigma_h^{k+1}u_h^{k+1}\|_{\mathbf{L}^2}^2 + 4\mu\tau\|u_h^{k+1}\|_{\mathbf{H}^1}^2 \le \frac{8\tau^2}{\chi}\|\nabla p_h^\sharp\|_{\mathbf{L}^2}^2 + \frac{\chi}{2}\|\sigma_h^{k+1}u_h^{k+1}\|_{\mathbf{L}^2}^2,$$

which by (3.4) implies that if τ small enough

$$\|\sigma_h^{k+1}u_h^{k+1}\|_{\mathbf{L}^2}^2 + 4\mu\tau\|u_h^{k+1}\|_{\mathbf{H}^1}^2 \le c\left(\|\sigma_h^0u_h^0\|_{\mathbf{L}^2}^2 + \|\sigma_h^1u_h^1\|_{\mathbf{L}^2}^2 + \frac{\tau^2}{\chi}\|\nabla p_h^0\|_{\mathbf{L}^2}^2 + \frac{\tau^2}{\chi}\|\nabla p_h^1\|_{\mathbf{L}^2}^2\right).$$

The estimate on the pressure is obtained *mutatis mutandis* the argument in the initialization step of the proof of Theorem 5.1. Hence

$$\begin{aligned}\|\sigma_h^{k+1}u_h^{k+1}\|_{\mathbf{L}^2}^2 + 4\mu\tau\|u_h^{k+1}\|_{\mathbf{H}^1}^2 + \frac{\tau^2}{\chi}\|\nabla p_h^{k+1}\|_{\mathbf{L}^2}^2 + \frac{\tau^2}{\chi}\|\nabla\delta p_h^{k+1}\|_{\mathbf{L}^2}^2 \le \\ c\left(\|\sigma_h^0u_h^0\|_{\mathbf{L}^2}^2 + \|\sigma_h^1u_h^1\|_{\mathbf{L}^2}^2 + \frac{\tau^2}{\chi}\|\nabla p_h^0\|_{\mathbf{L}^2}^2 + \frac{\tau^2}{\chi}\|\nabla p_h^1\|_{\mathbf{L}^2}^2\right), \quad k=1,2.\end{aligned}$$

(ii) *General Step:* For $k \ge 3$ we proceed as in the general step for the constant density case. Using (5.18) we obtain the estimate

$$\begin{aligned}(3-4M\tau)\|\sigma_h^{k+1}u_h^{k+1}\|_{\mathbf{L}^2}^2 - (4+8M\tau)\|\sigma_h^ku_h^k\|_{\mathbf{L}^2}^2 + (1-6M\tau)\|\sigma_h^{k-1}u_h^{k-1}\|_{\mathbf{L}^2}^2 + 2\|\sigma_h^{k+1}\delta u_h^{k+1}\|_{\mathbf{L}^2}^2 - 2\|\sigma_h^k\delta u_h^k\|_{\mathbf{L}^2}^2 \\ + \|\sigma_h^{k+1}\delta^2u_h^{k+1}\|_{\mathbf{L}^2}^2 + 4\mu\tau\|u_h^{k+1}\|_{\mathbf{H}^1}^2 + \frac{4\tau^2}{3\chi}\left[\|\nabla p_h^{k+1}\|_{\mathbf{L}^2}^2 - \|\nabla p_h^k\|_{\mathbf{L}^2}^2 + \|\nabla\delta p_h^k\|^2\right] \\ - \frac{4\tau^2}{3\chi}\|\nabla\delta^2p_h^{k+1}\|_{\mathbf{L}^2}^2 + \frac{8\tau^2}{9\chi}\langle\nabla\delta^2p_h^k, \nabla\delta p_h^{k+1}\rangle \le 0.\end{aligned}$$

Add and subtract to this inequality the terms $2\chi\|\delta u_h\|_{\mathbf{L}^2}^2$ taken at time steps t^{k+1} and t^k. Now, as in the constant density case, use the identity

$$\chi\left\|\delta u_h\right\|_{\mathbf{L}^2}^2 = \left\|\chi^{1/2}\delta u_h - \frac{2\tau}{3\chi^{1/2}}\nabla\delta^2p_h\right\|_{\mathbf{L}^2}^2 + \frac{4\tau^2}{9\chi}\left\|\nabla\delta^2p_h\right\|_{\mathbf{L}^2}^2,$$

to deduce

$$\begin{aligned}(3-4M\tau)\|\sigma_h^{k+1}u_h^{k+1}\|_{\mathbf{L}^2}^2 - (4+8M\tau)\|\sigma_h^ku_h^k\|_{\mathbf{L}^2}^2 + (1-6M\tau)\|\sigma_h^{k-1}u_h^{k-1}\|_{\mathbf{L}^2}^2 \\ + \|\sigma_h^{k+1}\delta^2u_h^{k+1}\|_{\mathbf{L}^2}^2 + 4\mu\tau\|u_h^{k+1}\|_{\mathbf{H}^1}^2 \\ + 2\left\|(\rho_h^{k+1}-\chi)^{1/2}\delta u_h^{k+1}\right\|_{\mathbf{L}^2}^2 - 2\left\|(\rho_h^k-\chi)^{1/2}\delta u_h^k\right\|_{\mathbf{L}^2}^2 \\ + 2\left\|\chi^{1/2}\delta u_h^{k+1} - \frac{2\tau}{3\chi^{1/2}}\nabla\delta^2p_h^{k+1}\right\|_{\mathbf{L}^2}^2 - 2\left\|\chi^{1/2}\delta u_h^k - \frac{2\tau}{3\chi^{1/2}}\nabla\delta^2p_h^k\right\|_{\mathbf{L}^2}^2 \\ + \frac{4\tau^2}{3\chi}\left[\|\nabla p_h^{k+1}\|_{\mathbf{L}^2}^2 - \|\nabla p_h^k\|_{\mathbf{L}^2}^2 + \|\nabla\delta p_h^k\|_{\mathbf{L}^2}^2\right] \\ - \frac{4\tau^2}{9\chi}\|\nabla\delta^2p_h^{k+1}\|_{\mathbf{L}^2}^2 - \frac{8\tau^2}{9\chi}\|\nabla\delta^2p_h^k\|_{\mathbf{L}^2}^2 + \frac{8\tau^2}{9\chi}\langle\nabla\delta^2p_h^k, \nabla\delta p_h^{k+1}\rangle \le 0,\end{aligned} \tag{5.19}$$

where we used assumption (3.4).

By assumption (3.4), the control on the last three pressure terms is obtained in a similar way as in the proof of Theorem 5.1, thus giving

$$-\frac{4\tau^2}{9\chi}\|\nabla\delta^2 p_h^{k+1}\|_{\mathbf{L}^2}^2 - \frac{8\tau^2}{9\chi}\|\nabla\delta^2 p_h^{k}\|_{\mathbf{L}^2}^2 + \frac{8\tau^2}{9\chi}\langle\nabla\delta^2 p_h^k, \nabla\delta p_h^{k+1}\rangle \geq \\ -\|\sigma_h^{k+1}\delta^2 u_h^{k+1}\|_{\mathbf{L}^2}^2 + \frac{4\tau^2}{9\chi}\left[\|\nabla\delta p_h^k\|_{\mathbf{L}^2}^2 - \|\nabla\delta p_h^{k-1}\|_{\mathbf{L}^2}^2\right].$$

Applying this estimate to (5.19) we arrive at the energy estimate

$$\begin{aligned}(3-4M\tau)\|\sigma_h^{k+1}u_h^{k+1}\|_{\mathbf{L}^2}^2 - (4+8M\tau)\|\sigma_h^k u_h^k\|_{\mathbf{L}^2}^2 + (1-6M\tau)\|\sigma_h^{k-1}u_h^{k-1}\|_{\mathbf{L}^2}^2 \\ + 4\mu\tau\|u_h^{k+1}\|_{\mathbf{H}^1}^2 \\ + 2\left\|(\rho_h^{k+1}-\chi)^{1/2}\delta u_h^{k+1}\right\|_{\mathbf{L}^2}^2 - 2\left\|(\rho_h^{k}-\chi)^{1/2}\delta u_h^{k}\right\|_{\mathbf{L}^2}^2 \\ + 2\left\|\chi^{1/2}\delta u_h^{k+1} - \frac{2\tau}{3\chi^{1/2}}\nabla\delta^2 p_h^{k+1}\right\|_{\mathbf{L}^2}^2 - 2\left\|\chi^{1/2}\delta u_h^{k} - \frac{2\tau}{3\chi^{1/2}}\nabla\delta^2 p_h^{k}\right\|_{\mathbf{L}^2}^2 \\ + \frac{4\tau^2}{3\chi}\left[\|\nabla p_h^{k+1}\|_{\mathbf{L}^2}^2 - \|\nabla p_h^k\|_{\mathbf{L}^2}^2 + \|\nabla\delta p_h^k\|_{\mathbf{L}^2}^2\right] \\ + \frac{4\tau^2}{9\chi}\left[\|\nabla\delta p_h^k\|_{\mathbf{L}^2}^2 - \|\nabla\delta p_h^{k-1}\|_{\mathbf{L}^2}^2\right] \leq 0.\end{aligned} \tag{5.20}$$

Introducing the notation

$$\begin{aligned} &A := 3-4M\tau, \quad B = -(4+8M\tau), \quad C = 1-6M\tau, \\ &a^k := \|\sigma_h^k u_h^k\|_{\mathbf{L}^2}^2, \quad k \geq 0, \\ &b^k := 4\mu\tau\|u_h^k\|_{\mathbf{L}^2}^2 + \frac{4\tau^2}{3\chi}\|\nabla\delta p_h^{k-1}\|_{\mathbf{L}^2}^2, \quad k\geq 1, \\ &d^k := 2\left\|\left(\rho_h^k-\chi\right)^{1/2}\delta u_h^k\right\|_{\mathbf{L}^2}^2 + 2\left\|\chi^{1/2}\delta u_h^k + \frac{2\tau}{3\chi^{1/2}}\nabla\delta^2 p_h^k\right\|_{\mathbf{L}^2}^2 + \frac{4\tau^2}{3\chi}\|\nabla p_h^k\|_{\mathbf{L}^2}^2 + \frac{4\tau^2}{9\chi}\|\nabla\delta p_h^{k-1}\|_{\mathbf{L}^2}^2, \quad k\geq 2,\end{aligned}$$

inequality (5.20) can be rewritten as

$$Aa^{k+1} + Ba^k + Ca^{k-1} \leq -\left(b^{k+1} + d^{k+1} - d^k\right), \quad k \geq 3.$$

Define $g^{k+1} := -(b^{k+1} + d^{k+1} - d^k)$. If τ is small enough, this three-term recursion inequality satisfies the assumptions of Proposition 5.2. The roots of the characteristic polynomial are

$$\begin{aligned} r_1 &:= \frac{2+4M\tau - \sqrt{1+38M\tau - 8M\tau^2}}{3-4M\tau} = \frac{1}{3}\left(1 - \frac{41M\tau}{3} + \mathcal{O}(\tau^2)\right), \\ r_2 &:= \frac{2+4M\tau + \sqrt{1+38M\tau - 8M\tau^2}}{3-4M\tau} = 1 + 9M\tau + \mathcal{O}(\tau^2).\end{aligned}$$

Both roots are positive, the first one is strictly less than one third, and the second is greater but close to one. Hence, for $\nu \geq 3$

$$a^\nu \leq c(a^1+a^2)(r_1^\nu + r_2^\nu) - \frac{1}{3-4M\tau}\sum_{l=3}^{\nu} r_1^{\nu-l}\sum_{s=3}^{l} r_2^{l-s}(b^s + d^s - d^{s-1}),$$

which, since τ is small, can be rewritten as

$$a^\nu + \frac{1}{3} b^\nu \le \mathcal{K}(1 + e^{cT})(a^1 + a^2) - \frac{1}{3 - 4M\tau} \sum_{l=3}^{\nu} r_1^{\nu - l} \sum_{s=3}^{l} r_2^{l-s}(d^s - d^{s-1}), \tag{5.21}$$

for some constants c and $\mathcal{K}$.

Notice that

$$\sum_{s=3}^{l} r_2^{l-s}(d^s - d^{s-1}) = d^l + (r_2 - 1) \sum_{s=3}^{l-1} r_2^{l-s-1} d^s.$$

Hence (5.21) implies

$$a^\nu + \frac{1}{3} b^\nu + \frac{1}{3} d^\nu \le \mathcal{K}(1 + e^{cT})(a^1 + a^2).$$

This inequality combined with the estimates obtained at the initialization step imply the result. □

Conjecture 5.1. *As numerical experiments show (see Section 6) the algorithm* (5.5)-(5.7)-(5.8) *performs as well as its constant density counterpart. This leads us to believe that the following error estimates hold:*

$$\|(\sigma u)_\tau - (\sigma_h u_h)_\tau\|_{\ell^\infty(\mathbf{L}^2)} \le c(\tau^2 + h^{l+1}),$$

and

$$\|u_\tau - (u_h)_\tau\|_{\ell^2(\mathbf{H}^1)} \le c(\tau + h^l).$$

The techniques presented here, together with those of [13] *may provide a proof of these facts.*

Remark 5.4. In full analogy with the constant density case, it is possible to construct a rotational version (see [20, 29]) of the algorithm introduced above by replacing the pressure update (5.8) by the following: Find $p_h^{k+1} \in M_h$ so that,

$$\left\langle p_h^{k+1}, r_h \right\rangle = \left\langle p_h^k + \phi_h^{k+1}, r_h \right\rangle + \mu \left\langle u_h^{k+1}, \nabla r_h \right\rangle. \tag{5.22}$$

The numerical experiments reported in Section 6 show that the algorithm (5.5)-(5.7)-(5.22) is stable and accurate. We have not been able to prove this fact yet.

6. Numerical Experiments

To test the accuracy of the second-order algorithm proposed in this paper, both in standard and rotational forms, we solve problem (1.1)-(1.2) using an analytical solution defined on the unit disk

$$\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}. \tag{6.1}$$

The exact solution is

$$\rho(\mathbf{r}, t) = 2 + r\cos(\theta - \sin t), \tag{6.2}$$

$$u(\mathbf{r}, t) = (-y, x)^\top \cos t, \tag{6.3}$$

$$p(\mathbf{r}, t) = \sin x \sin y \sin t, \tag{6.4}$$

and the corresponding right-hand side in the momentum equation is

$$f(\mathbf{r}, t) = \begin{pmatrix} (y \sin t - x \cos^2 t)\rho(\mathbf{r}, t) + \cos x \sin y \sin t \\ -(x \sin t + y \cos^2 t)\rho(\mathbf{r}, t) + \sin x \cos y \sin t \end{pmatrix}. \tag{6.5}$$

The computations are performed using the library `deal.II` (see [3, 2]). We use the $(\mathbb{Q}_2, \mathbb{Q}_2, \mathbb{Q}_1)$ approximation for the density, the velocity, and the pressure, respectively. We perform the accuracy

τ	ρ–L^2	**Rate**	u–$\mathbf{L}^2$	**Rate**	u–$\mathbf{H}^1$	**Rate**	p–L^2	**Rate**
0.1	9.15E-003	—	6.93E-003	—	3.29E-002	—	4.34E-002	—
0.05	1.27E-003	2.84	1.70E-003	2.03	9.93E-003	1.73	1.21E-002	1.84
0.03	2.10E-004	2.60	4.20E-004	2.02	3.20E-003	1.64	3.62E-003	1.74
0.01	4.18E-005	2.33	1.05E-004	2.00	1.11E-003	1.52	1.19E-003	1.60
0.01	8.65E-006	2.27	2.61E-005	2.00	3.63E-004	1.62	3.78E-004	1.66

TABLE 1. Error in Time for Standard Scheme

τ	ρ–L^2	**Rate**	u–$\mathbf{L}^2$	**Rate**	u–$\mathbf{H}^1$	**Rate**	p–L^2	**Rate**
0.1	3.70E-003	—	3.90E-003	—	1.59E-002	—	1.12E-002	—
0.05	6.38E-004	2.54	1.18E-003	1.73	4.89E-003	1.70	3.31E-003	1.76
0.03	1.35E-004	2.24	3.34E-004	1.82	1.43E-003	1.78	9.34E-004	1.83
0.01	3.21E-005	2.07	9.03E-005	1.89	4.03E-004	1.82	2.53E-004	1.88
0.01	7.85E-006	2.03	2.37E-005	1.93	1.12E-004	1.84	6.71E-005	1.92

TABLE 2. Error in Time for Rotational Scheme

tests with respect to τ on a mesh consisting of 5120 quadrangular cells. The dimensions of the vector spaces W_h, $\mathbf{X}_h$, and M_h are as follows:

$$\dim W_h = 20609, \tag{6.6}$$

$$\dim \mathbf{X}_h = 41218, \tag{6.7}$$

$$\dim M_h = 5185. \tag{6.8}$$

We measure the maximum over the time interval $[0, 10]$ of the errors measured in various norms. This mesh is chosen, so that the discretization error in space is significantly smaller than that induced by the time discretization. The convergence with respect to τ is verified in the range $5.10^{-3} \leq \tau \leq 1.10^{-1}$.

6.1. **The Standard Formulation.** We test the second-order standard formulation described in §5.1. The results are shown in Table 1. As expected, the error on the velocity and the density in the L^2-norm is of $\mathcal{O}(\tau^2)$ and the error on the velocity in the H^1-norm and on the pressure in the L^2-norm is of $\mathcal{O}(\tau)$.

6.2. **The Rotational Formulation.** Next we test the rotational version of the method which consists of using the pressure update (5.22), introduced in Remark 5.4, instead of (5.8). The results are shown in Table 2. We observe that all the errors are fully second-order with respect to τ. It is likely that there is a super-convergence effect due to the regularity of the domain. We recall that a similar super-converge effect is observed for the rotational variant of the pressure-correction algorithm for constant density flows (see [20]). We conjecture that in general domains the error on the velocity measured in the L^2-norm is $\mathcal{O}(\tau^2)$, and the error on the velocity in the H^1-norm and on the pressure in the L^2-norm is $\mathcal{O}(\tau^{3/2})$.

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